The work leading to this report was supported by funds provided by the National Council for Soviet and East European Research. The analysis and interpretations contained in the report are those of the author.
This report consists of two highly technical econometric modelling studies, both of which are described in more or less lay language in the Executive Summary. The studies are titled *Input Rationing and Bailouts Under Central Planning and Output Targets*, *Input Rationing and Inventories*, and are the product of the first phase of the COuncil contract identified on the face page, the Final Report of which will be available at a later date.
The first phase of the project resulted in the completion of two papers, both joint with Stephen M. Goldfeld of Princeton University. Both of these papers are direct consequences of the line of thought proposed in the original proposal.

The first of these, Input Rationing and Bailouts Under Central Planning, takes its point of departure from an earlier paper by Goldfeld and Quandt ("Budget Constraints, Bailouts and the Firm Under Central Planning," Journal of Comparative Economics, 12 (1988), 502-520) which attempted to formalize Kornai's notion of the soft budget constraint. The key idea here is that firms under central planning receive state subsidies when operating profits would turn out to be negative. The innovation of the earlier paper was to posit that managers may expend a specialized resource (a type of managerial labor) that can affect the amount of the "bailouts" that they receive; if you complain loudly enough, more will be given to you; however, the complaining itself does consume some resources. The analysis and the numerical simulations in the original paper confirmed Kornai's intuition that such a mechanism results in an increased demand for inputs, which presumably aggravates the extent of shortages.

The present paper combines this idea with another realistic modification that represents the prevalence of shortages, namely that one or more inputs that the manager needs may be rationed; that is to say, at the time that he requires inputs in certain
amounts, he may find that the entire desired amount is simply not available. A second innovation in the paper is the presence of another mechanism suggested by Kornai, namely that the expectation of future shortages may cause managers to purchase the input early, if at the earlier time the input should happen to be available more freely. Such anticipatory purchases have the effect of reducing or eliminating the possibility of a "no-shortage" situation at the earlier time, should that possibility exist in principle.

The paper first considers the case of a pure rationing model in which inputs at the second date are rationed but in which the manager can try to insure himself against an outage on the second date by buying early and carrying an inventory of the input at some cost to the enterprise. In a profit maximizing framework it turns out that under fairly general conditions the presence of rationing reduces the utilization of both the rationed and an unrationed input. In the next step, the paper combines the input rationing with the possibility of bailouts. It turns out that bailouts continue to have the effect that was found in the original Goldfeld-Quandt paper, namely that the presence of bailouts increase input demands over and above what they would have been in the absence of bailouts. Finally, it is interesting to compare the bailout-rationing case with the pure bailout case, i.e., when there is no rationing but bailouts are available. In this case only a somewhat weaker result can be established which says that there is surely some inventory cost high enough so that it will never pay the manager to engage in anticipatory purchases
of the potentially rationed input. It also can be shown that the total amount of input in the bailout-rationing case may be smaller or larger than in the pure bailout case. In conclusion, one can say that in general Kornai's conjecture continues to hold, but the presence of rationing, together with sufficiently high inventory carrying costs, will temper the manager's demand for inputs.

The second paper, Output Targets, Input Rationing and Inventories, combines the idea of input rationing with the notion that central planners prescribe an output target for the enterprise. In this paper the idea of bailouts is dispensed with. The key is that the manager attempts to minimize the expected deviation (squared) between his target and what he actually produces; once again, he is rationed but may compensate for the anticipated rationing in the present period by purchasing inputs in a previous period. This formulation leads to a flexible theoretical model in which it is possible to explore the consequences of (a) manipulable versus non-manipulable rationing, i.e., rationing in which the manager's input demand does or does not influence what he actually receives; (b) output or price uncertainty, i.e., cases in which the manager is not certain of what price he will be able to sell the output for or what precise output will actually materialize as a result of the enterprise's productive efforts; (c) the presence of defective inputs, i.e., a situation in which the inputs ordered by the manager are not necessarily all usable because of flaws in the upstream production process. In addition to these, a number of technical
variations in the assumptions are also entertained. Certain reasonable conclusions emerge from the analysis: the quantity of early input purchases increases if more inputs are needed altogether; this quantity increases as the penalty for missing the output target increases and decreases as the inventory cost increases; the more stringent the rationing is expected to be, the greater the early purchases of inputs will be. Essentially the same conclusions are obtained if some of the inputs are defective. We further examine the behavior of the solutions as other parameters of the problem are varied, such as the nature of the uncertainty concerning the rate of defective inputs and continue to find reasonable results such as the fact that a reduction in the defective rate reduces anticipatory input purchases. This indicates that planners have a tool for reducing inflationary pressures in the base period by concentrating efforts on raising the quality of production. Some of the shortage problems noted by Kornai may well be attributable to the low quality production that has been noted to exist in East European economies. An important concluding point to note is that even if there are no defectives, the total amount of inputs ordered in the type of case we consider, will in many instances actually exceed the total amount that is needed to produce the target level of output. Hence there tends to be an inherent "shortage bias" in the operations of enterprises.
Input Rationing and Bailouts Under Central Planning

by

Stephen M. Goldfeld
Richard E. Quandt

1. Introduction

It is generally believed that firms in a centrally planned economy operate under somewhat different ground rules than their counterparts in free economies. One of the most significant differences is the fact that in socialist economies persistent losses are not incompatible with the long-term survival of the firm.¹ The mechanisms by which survival is assured involve various forms of bailouts by the state and these institutions have been lucidly described by Kornai (1979, 1980, 1982, 1983) in connection with his theory of the soft budget constraint.

Relatively few formal treatments of the manager's decision problem in the presence of bailouts exist. Exceptions are Kornai and Weibull (1983), Hillman, Katz and Rosenberg (1987), Goldfeld and Quandt (1988a) and Ambrus-Lakatos and Csaba (1988). While these papers differ among one another in the formulation of the problem and in numerous details, they share the notion of a government bailout mechanism and the assumption that uncertainty is present in the manager's decision problem. For example, Hillman, Katz and Rosenberg assume that the price received for the output is uncertain and that the probability of receiving a bailout at all depends positively on total employment in the firm. Goldfeld and Quandt (1988a) develop a model that is consistent either with price uncertainty or

¹We are indebted to the National Science Foundation and the National Council for Soviet and East European Research for support.

¹Whether the recent introduction of bankruptcy procedures in countries such as Hungary will alter the validity of this statement remains to be seen.
uncertainty in the amount of output produced with given inputs and further assume that the amount of bailout received by the firm depends stochastically on the amount of specialized labor devoted to bailout-seeking. Both papers examine the effect of the bailout mechanism on the firm's demand for inputs within the framework of expected profit maximization; this is clearly one of the most important questions to examine in the light of Kornai's theory of shortage which predicts that the existence of a soft-budget constraint leads to a vastly increased demand for inputs and thus creates chronic shortages.

In the present paper we employ the important notions that (a) some inputs may not be available to the firm in unlimited quantities (see Portes (1969) for an earlier discussion of this) and (b) the firm may be able to hold inventories of the rationed input, i.e., purchasing the input in anticipation of future rationing. This feature of the model thus explicitly formalizes Kornai's view of anticipatory purchases which may be responsible for creating shortages in the present even if current input requirements would not lead to shortage. In Section 2 we introduce the model with rationing and inventories; in Section 3 we add the bailout mechanism to this model and in Section 4 we report some numerical experiments with the models. Section 5 contains some conclusions.

2. A Pure Rationing-Inventory Model

The firm is assumed to have a strictly concave production function with two inputs. The first input is subject to rationing, but the firm may attempt to deal with rationing by purchasing some amount of the first input

\footnote{In a quite different context, this question is also analyzed in Goldfeld and Quandt (1988b). That paper also considers the possibility of manipulable rationing, something that is ruled out in the present paper.}
at an earlier time. The amount of this input purchased at time 0 is $x_0$ and to hold it in inventory for one period incurs a unit carrying cost of $r$. One period later, the firm attempts to purchase an additional amount $x_1$ at the same price (and in this attempt the firm may or may not succeed) and also purchases an amount $x_2$ of the second input. Production then takes place instantaneously with whatever input amounts are at hand.\(^3\)

The amount of input 1 that is made available to the firm on date 1 is $z$ and we assume that $z$ is a random variable with density function $\rho(z)$ which is known to the firm. Given that the firm would like to obtain $x_1$ units of the first input at time 1, the actual amount obtained by the firm at that date is

$$x_1' = \min(x_1, z)$$

(2.1)

We finally assume that the output that materializes at date 1 is $f(x_0 + x_1', x_2) e^u$ where $u$ is normally distributed with mean zero and variance $\sigma^2$. Normalizing the output price to unity, the firm's profit is therefore given by\(^4\)

$$\pi(u) = f(x_0 + x_1', x_2) e^u - w_1(x_0 + x_1') - w_2 x_2 - rx_0 \quad \text{if } x_1' \leq z$$

$$\pi(u, z) = f(x_0 + z, x_2) e^u - w_1(x_0 + z) - w_2 x_2 - rx_0 \quad \text{if } x_1' > z$$

(2.2)

where $w_1$ and $w_2$ are the prices of the two inputs. Denoting the density of $u$ by $n(0, \sigma^2)$ and the cumulative distribution of $\rho(z)$ by $R(z)$, we can write expected profit as

\(^3\)Since no new information is assumed to be forthcoming, the decision on the variables $x_0, x_1$ and $x_2$ can be made at time 0. It should also be noted that when rationing is introduced, two things occur simultaneously: the availability of $x_1$ at date 1 is restricted and the price of input 1 is raised (because if acquired on date 0 its price is $w_1 + r$).

\(^4\)Alternatively, one may interpret the formulation as specifying a deterministic production function but a random price given by $e^u$. 

3

4
\[ E(\pi) = \left[ 1 - R(x_1) \right] \int_{-\infty}^{\infty} \pi_1(u)n(0, \sigma^2) du + \int_{0}^{\infty} \pi_2(u, z)n(0, \sigma^2) \rho(z) dz \]  

(2.3)

We assume that the manager is an expected profit maximizer. Profits, while not the sole objective, play an important part in the formulation of the objective function or the constraints of the problem in numerous static models (Donin (1976), Ireland and Law (1980), Ames (1965), Fortes (1963)). However, we do not pursue the formulation of such more complicated bonus functions for the manager because we seek to isolate the effects of rationing on the demand for inputs.

Expected profit maximization requires that we set the partial derivatives of \( E(\pi) \) with respect to \( x_0, x_1, x_2 \) equal to zero. The analytical results that can be obtained depend on whether the production function is a function of two inputs as specified in (2.2) or whether it is a function of a single (rationed) input, in which case it is written as \( f(x_0 + x_1) \) or \( f(x_0 + z) \), depending on the availability of the input at the later date. We shall distinguish the two cases in what follows.

**The Two-Input Case.** The first order conditions for a maximum are

\[ \frac{\partial E(\pi)}{\partial x_0} = \left[ 1 - R(x_1) \right] \left[ f_1(x_0 + x_1, x_2) e^{\sigma^2/2} - \omega_1 - r \right] + \]

\[ + \int_{0}^{x_1} \left[ f_1(x_0 + z, x_2) e^{\sigma^2/2} - \omega_1 - r \right] \rho(z) dz = 0 \]  

(2.4)

which may also be written as

\[ f_1(x_0 + x_1, x_2) e^{\sigma^2/2} + e^{\sigma^2/2} \int_{0}^{x_1} \left[ f_1(x_0 + z, x_2) - f_1(x_0 + x_1, x_2) \right] \rho(z) dz = \omega_1 + r \]  

(2.5)
\[ \frac{\delta E(\pi)}{\delta x_1} = \left[ 1 - R(x_1) \right] \left[ f_1(x_0 + x_1, x_2) e^{\sigma^2/2} - \psi \right] = 0 \]  \hspace{1cm} (2.6) \\
\frac{\delta E(\pi)}{\delta x_2} = f_2(x_0 + x_1, x_2) e^{\sigma^2/2} e^{\sigma^2/2} \int_{0}^{x_1} \left[ f_2(x_0 + z, x_2) - f_2(x_0 + x_1, x_2) \right] \rho(z) dz - \omega = 0 \]  \hspace{1cm} (2.7)

In the standard, competitive case in which no early purchases of the first input are considered, the first order conditions are \( f_1 e^{\sigma^2/2} = \psi \), \( f_2 e^{\sigma^2/2} = \omega \). Denote the solution to the first order conditions for the standard case by \( y^*, x_2^* \), where \( y \) takes the place of \( x_0 + x_1 \). Denote the solution to (2.5), (2.6) and (2.7) by \( x_0^{**}, x_1^{**}, x_2^{**} \). We then have the following

**Theorem 1.** If \( f_{12} > 0 \), then \( x_0^{**} + x_1^{**} < y^* \) and \( x_2^{**} < x_2^* \).

**Proof:** By the assumption that \( f_{12} > 0 \), the integral term in (2.7) is strictly negative for any value of \( x_0, x_1, x_2 \). Then (2.6) and (2.7) may be written as

\[ f_1(x_0 + x_1, x_2) e^{\sigma^2/2} = \psi \]  \hspace{1cm} (2.8)
\[ f_2(x_0 + x_1, x_2) e^{\sigma^2/2} = \omega + K(x_0, x_1, x_2) \]

with \( K(\ ) \) a positive function. This differs from the first order conditions for the standard case only by virtue of the presence of \( K \); the effect of this on \( x_0 + x_1 \) and on \( x_2 \) is equivalent to a positive perturbation of \( \omega \).

Taking total differentials (with \( d\psi = 0 \)) yields \( \delta(x_0 + x_1)/\delta \omega = -f_{12}/\Lambda \), \( \delta x_2/\delta \omega = f_{11}/\Lambda \), where \( \Lambda = f_{11} f_{22} - f_{12}^2 > 0 \) by concavity. Hence the conclusion follows.

The assumption \( f_{12} > 0 \) is plausible (e.g., it is automatically ensured for a concave and linearly homogeneous production function) and yields the conclusion that the effect of rationing is to reduce all input purchases, in spite

\[ \text{We omit the arguments of } f_1 \text{ and } f_2 \text{ when the omission creates no ambiguity.} \]
of the possibility of anticipatory buying of the rationed input.

As shown in Appendix A, there is no automatic assurance that the second order conditions are satisfied, although a sufficient condition is that

\[ e^{\sigma^2/2} \int_0^{x_1} f_{12}(x_0+z,x_2)p(z)dz \] is small. If the second order condition is satisfied, we obtain the following sensible comparative statics result:

**Theorem 2.** If the solution to (2.5), (2.6), (2.7) corresponds to a maximum, \( \partial x_0/\partial r < 0, \partial x_2/\partial r < 0. \)

**Proof:** Differentiating the first order conditions totally, and using the notation of Appendix A, we have

\[
\begin{bmatrix}
  a_{11} + b_{11} & a_{11} & a_{12} + b_{12} \\
  a_{11} & a_{11} & a_{12} \\
  a_{12} + b_{12} & a_{12} & a_{22} + b_{22}
\end{bmatrix}
\begin{bmatrix}
  dx_0 \\
  dx_1 \\
  dx_2
\end{bmatrix}
= \begin{bmatrix}
  dr \\
  0 \\
  0
\end{bmatrix}
\]

whence

\[ \frac{\partial x_0}{\partial r} = \frac{a_{11}a_{12} - a_{12} + a_{11}b_{22}}{\det(D)} < 0 \]

by the concavity of the production function and the second order condition \( \det(D) < 0 \). Similarly,

\[ \frac{\partial x_2}{\partial r} = -\frac{a_{11}b_{12}}{\det(D)} < 0 \]

No unambiguous effect on \( x_1 \) can be determined. On the other hand \( \partial (x_0 + x_1)/\partial r \) is less than zero. Varying \( w_1 \) and \( w_2 \) we can also determine analogously that \( \partial x_0/\partial w_1 < 0, \partial x_2/\partial w_1 < 0, \partial x_0/\partial w_2 < 0, \partial x_2/\partial w_2 < 0, \) with the effects on \( x_1 \) again indeterminate.

**The One-Input Case.** The first order conditions are straightforward adaptations of (2.4) and (2.6), keeping in mind that the production function

6
is now \( f(x_0 + x_1) \). We have
\[
\frac{\partial E(\pi)}{\partial x_0} = \left[ f'(x_0 + x_1) e^{\sigma^2/2} - w_1 \right] \left[ 1 - R(x_1) \right] + \int_0^{x_1} \left[ f'(x_0 + z) e^{\sigma^2/2} - w_1 \right] \rho(z) dz - r = 0 \quad (2.9)
\]
\[
\frac{\partial E(\pi)}{\partial x_1} = \left[ f'(x_0 + x_1) e^{\sigma^2/2} - w_1 \right] \left[ 1 - R(x_1) \right] = 0 \quad (2.10)
\]

Two consequences follow immediately.

**Theorem 3.** The solution for \( x_0 + x_1 \) is identical to the input purchased in the standard competitive case.

**Proof:** Obvious from (2.10)

**Theorem 4.** The second order conditions are satisfied.

**Proof:** Obvious from noting that the relevant Hessian matrix is the analog of D in Appendix A with its last row and column deleted.

**Remark 1.** In interpreting Theorem 3, it should be noted that \( x_1 \) is the amount ordered (demanded) for delivery at time 1 but is not in general the amount received. A better measure of the amount received is \( E(\min(x_1, z)) = x_1(1-R(x_1)) + \int_0^{x_1} z \rho(z) dz \) which is less than \( x_1 \).

**Remark 2.** As in the two-factor case, \( \partial x_0 / \partial r < 0 \). This follows directly from (2.9) and (2.10) which imply
\[
\frac{x_1}{0} \int \left[ f'(x_0 + z) e^{\sigma^2/2} - w_1 \right] \rho(z) dz = r
\]

Differentiating totally and using the fact from (2.10) that \( x_0 + x_1 \) is a constant,
\[
\frac{\partial x_0}{\partial r} = \left[ e^{\sigma^2/2} \int_0^{x_1} f''(x_0 + z) \rho(z) dz \right]^{-1} < 0
\]
from which it also follows that $\frac{dx_1}{dr} > 0$.

As a general conclusion we confirm the validity of Kornai's insight that the expectation of rationing induces anticipatory input purchases. While the magnitude of these purchases clearly depends on the inventory cost, we defer further comparative statics analyses to the numerical experiments reported in Section 4.

3. Rationing and Bailouts

We now introduce the notion of a bailout by the state in the event that the operating profit $\pi$ of the firm is negative (Goldfeld and Quandt (1988a)). The bailout subsidies are a percentage of the profit shortfall, with the percentage being an increasing function of an input $x_3$ which is taken to represent a specialized type of managerial labor, particularly suited to obtaining subsidies from the state authorities.\(^6\) This type of input does not contribute to the production of output and therefore does not appear in the production function which we continue to write as $f(x_0+x_1,x_2)$.

Analogously to (2.2) we can write the firm's operating profit as

$$\pi = \begin{cases} 
\pi_1(u) = f(x_0+x_1,x_2) e^{u} - w_1(x_0+x_1) - w_2 x_2 - r x_0 - w_3 x_3 & \text{if } x_1 \leq z \\
\pi_2(u,z) = f(x_0+z,x_2) e^{u} - w_1(x_0+z) - w_2 x_2 - r x_0 - w_3 x_3 & \text{if } x_1 > z 
\end{cases} \quad (3.1)$$

where $w_3$ is the unit price of $x_3$.\(^7\)

---

\(^6\)Ambrus-Lakatos and Csaba (1988) also consider the case in which positive operating profits are taxed by the state.

\(^7\)In Goldfeld and Quandt (1988a) we also explored a related model in which the firm first hires inputs other than $x_3$, produces output and hires $x_3$ after the resolution of output uncertainty if this is necessary. It can be argued that in the context of socialist planning this is a less plausible model; in any event its solution was similar to that of the precommitment model.
We now assume that the subsidy received by the firm when \( \pi \) is negative is equal to \( h(x_3v)\Pi \), where \( 0 \leq h \leq 1 \), \( h(0) = 0 \), \( h' > 0 \), and where \( v \) is a nonnegative random variable uncorrelated with \( u \), with density function \( g(v) \). This last assumption makes the efficiency of the \( x_3 \)-type labor depend on the state of nature, with good states corresponding to large values of \( v \).

The profit of the firm then is

\[
\pi = \begin{cases} 
\Pi & \text{if } \pi \geq 0 \\
\Pi(1-h(x_3v)) & \text{if } \pi < 0 
\end{cases} \tag{3.2}
\]

There are obviously four cases to consider, depending on whether \( \pi \geq 0 \) or \( \pi < 0 \) and whether \( x_1 \leq z \) or \( x_1 > z \). Defining \( C = (w_1+r)x_0+w_2x_2+w_3x_3 \), the operating profits in the four cases are given in Table 1 and the implied ranges of \( u \)-values in Table 2.

**Table 1. Operating Profits**

<table>
<thead>
<tr>
<th></th>
<th>( x_1 \leq z )</th>
<th>( x_1 &gt; z )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Pi \geq 0 )</td>
<td>( f(x_0+x_1,x_2)e^{u-C-w_1x_1} \geq 0 )</td>
<td>( f(x_0+z,x_2)e^{u-C-w_1z} \geq 0 )</td>
</tr>
<tr>
<td>( \Pi &lt; 0 )</td>
<td>( f(x_0+x_1,x_2)e^{u-C-w_1x_1} &lt; 0 )</td>
<td>( f(x_0+z,x_2)e^{u-C-w_1z} &lt; 0 )</td>
</tr>
</tbody>
</table>

**Table 2. Ranges of \( u \)**

<table>
<thead>
<tr>
<th></th>
<th>( x_1 \leq z )</th>
<th>( x_1 &gt; z )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Pi \geq 0 )</td>
<td>( u \leq \log \left[ \frac{C+w_1x_1}{f(x_0+x_1,x_2)} \right] )</td>
<td>( u \leq \log \left[ \frac{C+w_1z}{f(x_0+z,x_2)} \right] )</td>
</tr>
<tr>
<td>( \Pi &lt; 0 )</td>
<td>( u \leq \log \left[ \frac{C+w_1x_1}{f(x_0+x_1,x_2)} \right] )</td>
<td>( u \leq \log \left[ \frac{C+w_1z}{f(x_0+z,x_2)} \right] )</td>
</tr>
</tbody>
</table>
Defining \( \log \left[ \frac{(C + w_1 x_1)}{f(x_0 + x_1, x_2)} \right] \) as \( u_0 \) and \( \log \left[ \frac{(C + w_1 z)}{f(x_0 + z, x_2)} \right] \) as \( u_1(z) \), using (3.1), (3.2) and Table 2, we can write expected profit as

\[
E(\pi) = \left[ 1 - R(x_1) \right] \int_{u_0}^{u_0} \frac{\pi_1(u)n(0, \sigma^2)du}{\int_{v}^{v}} + \frac{\pi_1(u)n(0, \sigma^2)(1-h(x_3 v))g(v)dvdu}{\int_{v}^{v}} + \\
\int_{v}^{v} \frac{\pi_2(u, z)n(0, \sigma^2)\rho(z)dudz}{\int_{v}^{v}} + \\
\int_{v}^{v} \frac{\pi_2(u, z)n(0, \sigma^2)\rho(z)(1-h(x_3 v))g(v)dvdu}{\int_{v}^{v}} (3.3)
\]

This expression is simplified in (3.4) below. As indicated above, maximizing (3.3) with respect to \( x_0, x_1, x_2 \) and \( x_3 \) will, in general, cause the firm to pre-commit to hire some \( x_3 \) even if, ex post, it should turn out not to have been necessary.

It is straightforward to show that (3.3) is a generalization of both the rationing-inventory model discussed in Section 2 and of the pure bailout model with no rationing that was introduced in Goldfeld and Quandt (1988a). The absence of bailouts can be introduced into (3.3) by positing that the density of \( v \) is degenerate with a single masspoint at \( v=0 \). That causes the second and fourth terms of (3.3) to become zero. In addition, since we no longer need to integrate with respect to \( u \) over only a subset of the real
line, the lower limits of the u-integrals become $-\infty$. What remains is precisely (2.3). Alternatively, the absence of rationing is equivalent to $\rho(z)$ having a single masspoint at infinity; that causes the third and fourth terms of (3.3) to vanish and what remains is exactly Equation (3.4) in Goldfeld and Quandt (1988a).

As in the pure bailout case, only some analytic results are obtainable for the bailout-rationing case. To get these we first require (3.3) in a form more suitable for further manipulation. To achieve this, we substitute the normal density for $n(0, \sigma^2)$ everywhere and define

$$
\int \left[1 - h(x_3v)\right] g(v) dv = \psi(x_3)
$$

It was shown in Goldfeld and Quandt (1988a) that a sufficient condition for positive optimal $x_3$ in the pure bailout case is that $h$ satisfy a kind of Inada condition, i.e., that $h' \to -\infty$ as $x_3 \to 0$. We assume that this is the case here.

Since we wish to compare the present bailout rationing case with the pure bailout and the pure rationing cases, we denote the profits in these three cases by $\pi_{BR}$, $\pi_B$ and $\pi_R$ respectively. Straightforward integration yields

$$
E(\pi_{BR}) = \left[1 - R(x_1)\right] \left[f(x_0 + x_1, x_2)e^{\sigma^2/2} (1 - \psi_2) - (C + \omega_1) (1 - \psi_1)\right] + \\
+ \left[1 - R(x_1)\right] \psi(x_3) \left[f(x_0 + x_1, x_2)e^{\sigma^2/2} \psi_2 - (C + \omega_1) \psi_1\right] + \\
+ \int_0^{x_1} \rho(z) \left[f(x_0 + z, x_2)e^{\sigma^2/2} (1 - \psi_2(z)) - (C + \omega_1 z)(1 - \psi_1(z))\right] dz + \\
+ \psi(x_3) \int_0^{x_1} \rho(z) \left[f(x_0 + z, x_2)e^{\sigma^2/2} \psi_2(z) - (C + \omega_1 z) \psi_1(z)\right] dz
$$

(3.4)
where

\[ u_0/\sigma \]

\[ \psi_1 = \int \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{y^2}{2} \right\} dy \]

\[ (u_0-\sigma^2)/\sigma \]

\[ \psi_2 = \int \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{y^2}{2} \right\} dy \]

and where \( \psi_1(z) \), \( \psi_2(z) \) have the same form as \( \psi_1 \) and \( \psi_2 \), except that \( u_0 \) is replaced by \( u_1(z) \). It is also convenient to define \( \phi_1 \), \( \phi_2 \), \( \phi_1(z) \), \( \phi_2(z) \) as

\[ \phi_1 = \exp\left\{ -u_0^2/2\sigma^2 \right\}/\sqrt{2\pi}, \quad \phi_2=\exp\left\{ -(u_0-\sigma^2)/2\sigma^2 \right\}/\sqrt{2\pi}, \] with \( \phi_1(z) \), \( \phi_2(z) \) using \( u_1(z) \) instead of \( u_0 \). It is also convenient to obtain the partial derivatives of (3.4). For this we first need the following:

**Lemma.** \( f(x_0+x_1,x_2)e^{\sigma^2/2} \) \( \phi_2 = (C+w_1x_1)\phi_1 \) and \( f(x_0+z,x_2)e^{\sigma^2/2} \) \( \phi_2(z) = (C+w_1z)\phi_1(z) \).

**Proof:** Straightforward substitution yields \( \phi_1/\phi_2 = \exp\left\{ u_0+\sigma^2/2 \right\} \) and \( \phi_1(z)/\phi_2(z) = \exp\left\{ -u_1(z)+\sigma^2/2 \right\} \). Substituting \( \log[f(x_0+x_1,x_2)/(C+w_1x_1)] \) for \( u_0 \) and \( \log[f(x_0+z,x_2)/(C+w_1z)] \) for \( u_1(z) \) yields the result.

The partial derivatives then are

\[ \frac{\partial F(\pi)}{\partial x_0} = (1-R(x_1)) \left[ f_1(x_0+x_1,x_2)e^{\sigma^2/2}\left[ 1-(1-\psi)\psi_2 \right] - (w_1+r)\left[ 1-(1-\psi)\psi_1 \right] \right] \]

\[ + \int_0^{\infty} \rho(z) \left[ f_1(x_0+z,x_2)e^{\sigma^2/2}\left[ 1-(1-\psi)\psi_2(z) \right] - (w_1+r)\left[ 1-(1-\psi)\psi_1(z) \right] \right] dz \] (3.5)
\[
\frac{\partial E(\pi)}{\partial x_1} = (1-R(x_1))\left[ f_1(x_0+x_1,x_2)e^{\sigma^2/2-w_1} - (1-\psi)f_1(x_0+x_1,x_2)e^{\sigma^2/2+2-w_1+s_1} \right] \quad (3.6)
\]

\[
\frac{\partial E(\pi)}{\partial x_2} = (1-R(x_1))\left[ f_2(x_0+x_1,x_2)e^{\sigma^2/2}\left[ 1-(1-\psi)^2 \right] - w_2\left[ 1-(1-\psi)^1 \right] \right]
\]

\[
+ \int_0^{\infty} \rho(z)\left[ f_2(x_0+z,x_2)e^{\sigma^2/2}\left[ 1-(1-\psi)^2(z) \right] - w_2\left[ 1-(1-\psi)^1(z) \right] \right] dz \quad (3.7)
\]

We are now in a position to obtain some analytic results for the bailout-rationing case. Since bailouts tend to increase the use of inputs when added to the competitive model, it is natural to inquire if the same effect results from adding bailouts to the rationing model. We thus first compare the bailout-rationing model to the rationing model. We subsequently compare the bailout-rationing model to the pure bailout model.

**Bailout-Rationing versus Pure Rationing.** The analytic results that can be obtained depend on whether we are dealing with the two-factor case in Equations (3.5)-(3.7) or with a one-factor case in which the production function is \( f(x_0+x_1) \). For the former case we have

**Theorem 5.** At the point at which \( E(\pi_R) \) attains a maximum,

\[
\frac{\partial E(\pi_R)}{\partial x_1} > 0.
\]

**Proof:** Comparing (3.6) with (2.6) shows that where \( E(\pi_R) \) attains a maximum, \( \frac{\partial E(\pi_R)}{\partial x_1} \) can be written as

\[
\frac{\partial E(\pi_R)}{\partial x_1} = -(1-\psi)f_1(x_0+x_1,x_2)e^{\sigma^2/2+s_2-w_1+s_1} \quad (3.8)
\]

At the point of maximum \( E(\pi_R) \) we must have \( f_1e^{\sigma^2/2} - w_1 = 0 \). But \( s_2 < s_1 \) and hence the bracket in (3.8) is negative and by the definition of \( h(\cdot), \psi < 1 \), which completes the proof.
Remark 3. The analytic results are clearly local in nature. What Theorem 5 says is that the manager of the firm, starting from the point of maximum $E(\pi_1)$ and acting as a steepest-ascent maximizer, will want to increase his employment of $x_1$ (i.e., the first input demanded on date 1) in response to the availability of bailouts. This result is then analogous to that in Goldfeld and Quandt (1988a) in which the (pure) bailout model was compared to the standard competitive model. While analytic results are not available for $x_0$ and $x_2$, numerical results are reported in Section 4 and consistent with the analytical result for $x_1$.

A related but different proposition can be proved for the one-factor case. For the purpose of this proposition, $f_1(x_0+x_1,x_2)$ is (temporarily) replaced by $f'(x_0+x_1)$. We have

Theorem 6. If $R(x_1) \neq 1$, the solution of the first order condition of (3.4) for $x_0+x_1$ is greater than the solution of the first order conditions of (2.3).

Proof: Equation (3.6) can now be rewritten as

$$\frac{\partial E(\pi_1)}{\partial x_1} = \left[1-R(x_1)\right] \left[f'(x_0+x_1)e^{\sigma^2/2} - \omega_1\right] +$$

$$+ \left[1-R(x_1)\right] \left[\psi(x_3)-1\right] \left[f'(x_0+x_1)e^{\sigma^2/2+2-\omega_1}\right] = 0$$

and thus

$$f'(x_0+x_1)e^{\sigma^2/2} = kw_1 \tag{3.9}$$

where $k = \left[1-(1-\psi(x_3))\omega_1\right]\left[1-(1-\psi(x_3))\omega_2\right]^{-1} < 1$ by the definition of $\omega_1$ and $\omega_2$.

But the corresponding first order condition (2.10) implies
f'(x_0 + x_1)e^{\sigma^2/2} = w_1; it follows from concavity that the solution for x_0 + x_1 in the present case must be greater.

**Remark 4.** The condition R(x_1)\neq 1 is understandable, since if it does not hold, infinitely many x_0 + x_1 values are compatible with satisfying the first order conditions. It may also be noted that Theorem 6 is a global rather than local theorem.

**Bailout-Rationing versus Pure Bailout.** We first rewrite equations (3.5) - (3.7) by evaluating them at the point at which E(\pi_0) attains a maximum, i.e., at which \( \frac{\partial E(\pi_0)}{\partial x_i} (i=1,2,3) \) is zero and x_0 is also zero. This yields

\[
\frac{\partial E(\pi_{DR})}{\partial x_0} = -r(1-R(x_1)){(1-(1-\psi)\psi_1)} +
\]

\[
(x_1)\int_0^1 \rho(z)\left[f_1(x_0+z,x_2)e^{\sigma^2/2}[1-(1-\psi)\phi_2(z)] - (w_1+r)(1-(1-\psi)\psi_1(z))\right]dz (3.10)
\]

\[
\frac{\partial E(\pi_{DR})}{\partial x_1} = 0 \quad (3.11)
\]

\[
\frac{\partial E(\pi_{DR})}{\partial x_2} = \int_0^1 \rho(z)\left[f_2(x_0+z,x_2)e^{\sigma^2/2}[1-(1-\psi)\phi_2(z)] - w_2(1-(1-\psi)\psi_1(z))\right]dz (3.12)
\]

The primary question of interest concerns the relationship of the optimal x_0 + x_1 in the bailout-rationing case to the optimal x_1 in the pure bailout case (in which there is no x_0). Prior to doing this, it is useful to examine the behavior of (3.10). It is straightforward to note that if \( \rho(z) = 0 \) except for a masspoint at infinity, then at the point at which E(\pi_0) is a maximum we have \( \frac{\partial E(\pi_{DR})}{\partial x_0} = -r(1-R(x_1))(1-(1-\psi)\psi_1) < 0 \).
This is the case in which the rationing density is such that rationing never takes place. The result is exactly what has to be true. Since in the pure bailout case there is no distinction between $x_0$ and $x_1$, that case determines only an optimal level of $x_0 + x_1$. What the result expresses is that if we were arbitrarily to decide to have a positive amount of $x_0$, this could not be optimal for the bailout-rationing case. A related line of argument leads to

**Theorem 7.** For any $\rho(z)$ there exists a value of $r$ sufficiently large so that $\partial E(\pi \rho_0)/\partial x_0 < 0$ at the point at which $E(\pi_0)$ is at a maximum.

**Proof:** If the integral in (3.10) is negative, the conclusion follows. If the integral is positive for some value of $r$, then we can always find an $r^* > r$ such that the integral is zero. But then, for this $r^*$, the conclusion follows.

Clearly this establishes the reasonable case that for some level of inventory storage costs it cannot be optimal to hire inputs in the amounts that are optimal for the pure bailout case and have some of the first input purchased at the earlier date. However, it is important to note that both Theorem 7 gives only a local result in the neighborhood of the point corresponding to a maximum of $E(\pi_0)$.

We now return to the relationship of the optimal $x_0 + x_1$ in the bailout-rationing case to the optimal $x_1$ in the pure bailout case. There is, however, no unambiguous relationship between the two solutions. We again evaluate (3.5) at the point of maximum $E(\pi_0)$. The pure bailout solution determines a value for $x_0 + x_1$ in which $x_0$ is zero, and so this is the point at which the evaluation occurs.

$$\frac{\partial E(\pi \rho_0)}{\partial x_0} = -r(1-R(x_1))[1-(1-\psi^*)_1] + \int_0^{x_1} \rho(z)[T_1(z)-T_2(z)]dz \quad (3.13)$$
where
\[ T_1(z) = f_1(x_0+z,x_2)e^{\gamma^2/2} \left[ 1-(1-\psi)\phi_2(z) \right] \]
\[ T_2(z) = (\psi_1+r)\left[ 1-(1-\psi)\phi_1(z) \right] \]

Then, if the Inada condition holds for the production function, i.e., if
\[ \lim_{x_0+z \to 0} f_1 = 0 \]
as is true for the Cobb-Douglas production function, \( T_1(z) \) is arbitrarily large for \( z \) near zero and the integrand is positive. If \( r \) is sufficiently large, the integrand in (3.13) is negative for \( z \) not near zero. The sign of the integrand then clearly depends on the behavior of \( \rho(z) \). The conclusion that emerges is that at the optimum for \( E(\pi) \), while the slope of \( E(\pi_{Bj}) \) in the \( x_1 \) direction is zero, the slope in the \( x_0 \) direction may be positive or negative, hence locally \( x_0+x_1 \) may either increase or decrease as we seek better \( E(\pi) \) values. We explore this further in the next section.

4. Some Numerical Results

The previous sections concentrated on some general qualitative results in the presence of input rationing. However, the theorems prove only local results; i.e., they predict in what direction the input purchases of an enterprise will move (initially) if the enterprise manager seeks optima by the steepest ascent method. Numerical experiments, in which we assume some specific functional forms and parameter values, can help to reveal the global characteristics of the problems at hand. Of course, the results of such experiments are only illustrative; however, they can shed substantial light on the underlying relations and they provide a tool with which additional scenarios can also be examined.\(^8\)

\(^8\) Maximization of the various expected profit functions was accomplished with GQOPT.
The Basic Setup. We assume that the deterministic part of the production function is Cobb-Douglas so that

\[ f(x_0 + x_1, x_2) = (x_0 + x_1)^a x_2^b \quad (4.1) \]

Output is \( f(x_0 + x_1, x_2) e^u \) or \( f(x_0 + z_1, x_2) e^u \), depending on whether \( x_1 \leq z \) or not, where \( u \) is normally distributed with mean zero and variance \( \sigma^2 \). We assume that \( \rho(z) \) is a Weibull density, i.e., that \( R(z) = 1 - \exp\left(-\frac{z}{b}\right)^c \). We further assume that

\[ h(x_3, \nu) = 1 - \exp\left(-a_1 x_3^{-1} \right) \quad (4.2) \]

where \( a_1 > 0 \), \( 0 < a_2 < 1 \) and where \( \nu \) is distributed exponentially with \( g(\nu) = \gamma e^{-\gamma \nu} \). It is easy to verify that this choice of \( h \) satisfies the Inada condition imposed in Section 3.

We vary scenarios by choosing alternative sets of specific values for the parameters. For each scenario we obtain the solution for (a) the standard competitive model (C), (i.e., the expected profit maximizing solution in the absence of bailouts and rationing), (b) the pure rationing model (R), (c) the pure bailout model (B), and (d) the bailout-rationing model (BR). In order to limit the experiments, we did not vary all the parameters and specifically assume in most experiments that \( w_1 = w_2 = 0.4 \) and that \( \alpha = \beta = 0.4 \). The consequence of these assumptions is that the optimal solutions for \( x_1 \) and \( x_2 \) are identical to each other in cases C and B.

Our base case is characterized by the following parameter values:

- \( w_1 = w_2 = w_3 = 0.4 \)
- \( \alpha = \beta = 0.4 \)
- \( \sigma_1^2 = 0.1 \)
- \( a_1 = 5.0 \)
- \( a_2 = 0.5 \)
- \( \gamma = 0.5 \)
- \( r = 0.01 \)
- \( b = 2.0 \)
- \( c = 3.0 \)

and \( \sigma_w \) of the Weibull density are 1.79 and 0.65 respectively.

An alternative would be to assume that \( g(\nu) \) is a Gamma density. Experimentation with this assumption by Goldfeld and Quandt (1988a) showed that this had only a modest effect on the results.
Table 1 displays a number of alternative scenarios; we identify each case in terms of those parameters the values of which differ from the base case. Some parameter variations leave the results for some of the models unchanged from the base case and for each case we display results only for those models where a change does occur.

We first note from Table 1 that the comparison of C and R agrees with the prediction of Theorem 1 in that \( x_0 + x_1 \) and \( x_2 \) in the pure rationing case are always smaller than the corresponding input quantities in the standard competitive case. Theorem 2 predicts that an increase in \( r \) will tend to reduce \( x_0 \) and \( x_2 \); the comparison of Cases 6 and 9 shows that this in fact happens for Model R (as well as for Model BR). The corollary to Theorem 2 also predicts the comparative statics changes for variations in \( w_1, w_2 \); the observed changes from Case 6 to Cases 7 and 8 respectively are in the predicted direction. The theorem and its corollary are inconclusive with respect to \( x_1 \); the numerical calculations show that \( x_1 \) responds in the same direction as \( x_0 \) and \( x_2 \).

Theorem 5 suggests that the BR case will have a higher \( x_1 \) than Model R and this is uniformly the case. The argument at the end of the last section also indicates that no unambiguous comparison can be made between the optimal \( x_0 + x_1 \) in Model BR and the optimal \( x_1 \) in Model B. In fact, the numerical results show that the comparison can go either way: in most cases \( x_0 + x_1 \) for Model B is greater than for Model BR, but in Case 5 the comparison goes the other way.

We also note that a reduction in \( a_1 \) reduces the bailout-securing effectiveness of \( x_3 \)-type labor and diminishes input demands (Case 2 compared to Case 1).\(^{10}\) A reduction in uncertainty (Case 6 vs. Case 1) reduces all input demands but has its most drastic effects in models in-

\(^{10}\) However, Goldfeld and Quandt (1988a) found that the demand for \( x_3 \) is not monotone in \( a_1 \): as \( a_1 \) increases, \( x_3 \) first increases and then falls.
involving bailout. The soft budget constraint thus appears to have its most pronounced effects when output (or price) uncertainty is great. An increase in $\gamma$ alters the uncertainty concerning the effectiveness of $x_3$.

Since $\gamma(x_3) = \left[\frac{a_1}{\gamma}x_3^{a_2} + 1\right]^{-1}$ in the present case, the effect of variations in $\gamma$ should be the opposite of those caused by variations in $a_1$. This is in fact the case for both Models B and BR (compare Cases 3 and 4). Finally, variations in $w_1$ and $w_2$ (Cases 6, 7, 8) have the expected effects.

Table 2 displays a few cases when there is only one input and the production function is $f(x_0 + x_1)$. The behavior here is completely similar to the two factor case except that, in addition, Theorem 3 holds as can be seen from comparing $x_0 + x_1$ for Models C and R. We also verify that the comparison of $x_0 + x_1$ for Models B and BR can go either way (see, for example, Cases 1 and 5).
Table 1. Results for Two-Factor Cases

<table>
<thead>
<tr>
<th>Case</th>
<th>$x_0$</th>
<th>$x_1$</th>
<th>$x_0+x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$E(x_1)$</th>
<th>$E(\pi)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Base Case:</td>
<td>C</td>
<td>0</td>
<td>2.09</td>
<td>2.09</td>
<td>2.09</td>
<td>0</td>
<td>--</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>0</td>
<td>8.93</td>
<td>8.93</td>
<td>8.93</td>
<td>0.4018</td>
<td>--</td>
</tr>
<tr>
<td></td>
<td>R</td>
<td>0.76</td>
<td>1.23</td>
<td>2.00</td>
<td>1.97</td>
<td>0</td>
<td>1.17</td>
</tr>
<tr>
<td></td>
<td>BR</td>
<td>6.25</td>
<td>1.91</td>
<td>8.16</td>
<td>8.01</td>
<td>0.3561</td>
<td>1.58</td>
</tr>
<tr>
<td>2. $a_1 = 1.0$</td>
<td>B</td>
<td>0</td>
<td>2.82</td>
<td>2.32</td>
<td>2.82</td>
<td>0.0619</td>
<td>--</td>
</tr>
<tr>
<td></td>
<td>BR</td>
<td>1.29</td>
<td>1.35</td>
<td>2.64</td>
<td>2.59</td>
<td>0.0527</td>
<td>1.25</td>
</tr>
<tr>
<td>3. $\sigma_1^2 = 0.025$</td>
<td>C</td>
<td>0</td>
<td>1.63</td>
<td>1.63</td>
<td>1.63</td>
<td>0</td>
<td>--</td>
</tr>
<tr>
<td>$b = 1.0$</td>
<td>B</td>
<td>0</td>
<td>1.91</td>
<td>1.91</td>
<td>1.91</td>
<td>0.0098</td>
<td>--</td>
</tr>
<tr>
<td>$c = 1.0$</td>
<td>R</td>
<td>1.19</td>
<td>0.36</td>
<td>1.55</td>
<td>1.53</td>
<td>0</td>
<td>0.30</td>
</tr>
<tr>
<td></td>
<td>BR</td>
<td>1.41</td>
<td>0.40</td>
<td>1.81</td>
<td>1.78</td>
<td>0.0089</td>
<td>0.33</td>
</tr>
<tr>
<td>4. $\sigma_1^2 = 0.025$</td>
<td>C</td>
<td>0</td>
<td>1.63</td>
<td>1.63</td>
<td>1.63</td>
<td>0</td>
<td>--</td>
</tr>
<tr>
<td>$\gamma = 0.7$</td>
<td>B</td>
<td>0</td>
<td>1.83</td>
<td>1.83</td>
<td>1.83</td>
<td>0.0081</td>
<td>--</td>
</tr>
<tr>
<td>$b = 1.0$</td>
<td>R</td>
<td>1.19</td>
<td>0.36</td>
<td>1.55</td>
<td>1.53</td>
<td>0</td>
<td>0.30</td>
</tr>
<tr>
<td>$c = 1.0$</td>
<td>BR</td>
<td>1.35</td>
<td>0.39</td>
<td>1.74</td>
<td>1.71</td>
<td>0.0072</td>
<td>0.32</td>
</tr>
<tr>
<td>5. $a_1 = 0.2$</td>
<td>B</td>
<td>0</td>
<td>2.12</td>
<td>2.12</td>
<td>2.12</td>
<td>0.0030</td>
<td>--</td>
</tr>
<tr>
<td>$b = 0.8$</td>
<td>R</td>
<td>1.23</td>
<td>2.79</td>
<td>4.02</td>
<td>1.98</td>
<td>0</td>
<td>0.71</td>
</tr>
<tr>
<td>$c = 3.1$</td>
<td>BR</td>
<td>1.26</td>
<td>2.79</td>
<td>4.05</td>
<td>2.01</td>
<td>0.0027</td>
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</tr>
<tr>
<td>6. $\sigma_1^2 = 0.025$</td>
<td>C</td>
<td>0</td>
<td>1.63</td>
<td>1.63</td>
<td>1.63</td>
<td>0</td>
<td>--</td>
</tr>
<tr>
<td>$b = 1.0$</td>
<td>B</td>
<td>0</td>
<td>1.91</td>
<td>1.91</td>
<td>1.91</td>
<td>0.0098</td>
<td>--</td>
</tr>
<tr>
<td>$c = 1.0$</td>
<td>R</td>
<td>0.42</td>
<td>1.14</td>
<td>1.56</td>
<td>1.54</td>
<td>0</td>
<td>1.14</td>
</tr>
<tr>
<td></td>
<td>BR</td>
<td>0.61</td>
<td>1.20</td>
<td>1.81</td>
<td>1.77</td>
<td>0.0083</td>
<td>1.14</td>
</tr>
<tr>
<td>7. $\sigma_1 = 0.025$</td>
<td>C</td>
<td>0</td>
<td>2.72</td>
<td>2.72</td>
<td>2.38</td>
<td>0</td>
<td>--</td>
</tr>
<tr>
<td>$\omega_1 = 0.35$</td>
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<td>3.27</td>
<td>3.27</td>
<td>2.86</td>
<td>0.0154</td>
<td>--</td>
</tr>
<tr>
<td>$c = 1.0$</td>
<td>R</td>
<td>1.20</td>
<td>1.38</td>
<td>2.58</td>
<td>2.21</td>
<td>0</td>
<td>1.33</td>
</tr>
<tr>
<td></td>
<td>BR</td>
<td>1.59</td>
<td>1.46</td>
<td>3.05</td>
<td>2.62</td>
<td>0.0131</td>
<td>1.33</td>
</tr>
<tr>
<td>8. $\sigma_1^2 = 0.025$</td>
<td>C</td>
<td>0</td>
<td>2.38</td>
<td>2.38</td>
<td>2.72</td>
<td>0</td>
<td>--</td>
</tr>
<tr>
<td>$\omega_2 = 0.35$</td>
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<td>0</td>
<td>2.86</td>
<td>2.86</td>
<td>3.27</td>
<td>0.0154</td>
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</tr>
<tr>
<td>$c = 1.2$</td>
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<td>0.99</td>
<td>1.28</td>
<td>2.27</td>
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<td>0</td>
<td>1.21</td>
</tr>
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<td>1.30</td>
<td>2.64</td>
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<td>0.0133</td>
<td>1.22</td>
</tr>
<tr>
<td>9. $\sigma_1^2 = 0.025$</td>
<td>R</td>
<td>0.29</td>
<td>1.24</td>
<td>1.53</td>
<td>1.49</td>
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<td>1.17</td>
</tr>
<tr>
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<td>BR</td>
<td>0.45</td>
<td>1.30</td>
<td>1.75</td>
<td>1.71</td>
<td>0.0075</td>
<td>1.22</td>
</tr>
</tbody>
</table>

21
Table 2. Results for One-Factor Cases

<table>
<thead>
<tr>
<th>Case</th>
<th>x₀</th>
<th>x₁</th>
<th>x₀+x₁</th>
<th>x₃</th>
<th>E(x₁)</th>
<th>E(π)</th>
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</thead>
<tbody>
<tr>
<td>1. Base Case</td>
<td>C</td>
<td>0</td>
<td>4.62</td>
<td>4.62</td>
<td>0</td>
<td>--</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>0</td>
<td>5.07</td>
<td>5.07</td>
<td>0.0132</td>
<td>--</td>
</tr>
<tr>
<td></td>
<td>R</td>
<td>2.78</td>
<td>1.84</td>
<td>4.62</td>
<td>0</td>
<td>1.55</td>
</tr>
<tr>
<td></td>
<td>BR</td>
<td>3.13</td>
<td>1.96</td>
<td>5.09</td>
<td>0.0122</td>
<td>1.60</td>
</tr>
<tr>
<td>2. b = 1.0</td>
<td>R</td>
<td>3.73</td>
<td>0.89</td>
<td>4.62</td>
<td>0</td>
<td>0.59</td>
</tr>
<tr>
<td>c = 1.0</td>
<td>BR</td>
<td>4.10</td>
<td>1.01</td>
<td>5.11</td>
<td>0.0126</td>
<td>0.64</td>
</tr>
<tr>
<td>3. r = 0.015</td>
<td>R</td>
<td>2.56</td>
<td>2.06</td>
<td>4.62</td>
<td>0</td>
<td>1.64</td>
</tr>
<tr>
<td></td>
<td>BR</td>
<td>2.89</td>
<td>2.21</td>
<td>5.10</td>
<td>0.0115</td>
<td>1.68</td>
</tr>
<tr>
<td>4. w₃ = 0.3</td>
<td>B</td>
<td>0</td>
<td>5.12</td>
<td>5.12</td>
<td>0.0179</td>
<td>--</td>
</tr>
<tr>
<td></td>
<td>R</td>
<td>2.78</td>
<td>1.84</td>
<td>4.62</td>
<td>0</td>
<td>1.55</td>
</tr>
<tr>
<td></td>
<td>BR</td>
<td>3.17</td>
<td>1.98</td>
<td>5.15</td>
<td>0.0165</td>
<td>1.61</td>
</tr>
<tr>
<td>5. b = 1.0</td>
<td>B</td>
<td>0</td>
<td>5.07</td>
<td>5.07</td>
<td>0.0132</td>
<td>--</td>
</tr>
<tr>
<td>c = 1.0</td>
<td>R</td>
<td>0.72</td>
<td>3.89</td>
<td>4.62</td>
<td>0</td>
<td>--</td>
</tr>
<tr>
<td>r = 0.2</td>
<td>BR</td>
<td>0.76</td>
<td>4.18</td>
<td>4.94</td>
<td>0.0020</td>
<td>--</td>
</tr>
</tbody>
</table>
We have investigated the behavior of the enterprise under a regime of central planning when two important features are present: (1) The firm receives bailouts from the state when operating profits are negative and the enterprise devotes some resources to securing subsidies, and (2) an input is rationed, but the enterprise can mitigate the effect of rationing by purchasing early and incurring some inventory cost. Our analysis compared the input demands of the firm when the bailout regime or the rationing regime or both are imposed. The propositions that we proved were of a local nature; i.e., they predict the direction in which the enterprise will move if it is employing a steepest-ascent method of optimization but numerical experiments suggest that these propositions hold more globally.

There are several additional complicating factors that need to be addressed in future analysis. (1) What will be the impact in the bailout-rationing context of a regime in which positive profits are taxed by the state, perhaps at a rate that may be influenced by the efforts of the firm to secure tax-abatement, as suggested by Ambrus-Lakatos and Csaba (1988)? (2) What will be the impact of a more general system of rationing in which all inputs are rationed? (3) What is the impact of a manipulable rationing scheme in which effort may be devoted by the firm to secure favorable rationing parameters? Finally, (4) what is the impact of output targets (central plan targets) being imposed at the same time that one or more inputs are being rationed? We hope to address some of these questions in future work.
Appendix A. Second Order Conditions for the Pure Rationing Model

Invoking the first order conditions, the matrix of second partial derivatives can be shown to be

\[ D = \begin{bmatrix} a_{11} + b_{11} & a_{11} & a_{12} + b_{12} \\ a_{11} & a_{11} & a_{12} \\ a_{12} + b_{12} & a_{12} & a_{22} + b_{22} \end{bmatrix} \]  \hspace{1cm} (A.1)

where

\[ a_{ij} = \left[ 1 - R(x_1) \right] f_{ij}(x_0 + x_1, x_2) \sigma^2 / 2 \]

\[ b_{ij} = e^{\sigma^2 / 2} \int_{0}^{x_1} f_{ij}(x_0 + z, x_2) \rho(z) dz \]

and \( a_{ii} < 0, b_{ii} < 0 \) (\( i = 1, 2 \)) and \( a_{12}, b_{12} > 0 \) if \( f_{12} > 0 \).

The second order conditions require, among others, that the determinant of \( D \) be negative. We have

\[ \det(D) = b_{11} \left( a_{11} a_{22} - a_{12}^2 \right) + a_{11} \left( b_{11} b_{22} - b_{12}^2 \right) \]  \hspace{1cm} (A.2)

and a sufficient condition for \( \det(D) \) to be negative is that \( b_{12} \) be sufficiently small. Although this condition is not necessary, there is no a priori assurance that the second order condition is satisfied.
References


____ (1980), Economics of Shortage, Amsterdam: North-Holland.


ABSTRACT

Output Targets, Input Rationing and Inventories

by

Stephen M. Goldfeld
Richard E. Quandt

The paper introduces and elaborates a model of the enterprise in a centrally planned economy. Inputs can be purchased on two dates. On the first of these dates the input is available without limitation, but on the second of these dates it may be rationed. The enterprise, which has to produce so as to meet an output target on the second date, may insure itself against not having enough inputs by purchasing early at the cost of some carrying charges. The model is, in a sense, a formalization of a suggestion by Kornai that the anticipation of future rationing may induce anticipatory input purchases by the enterprise.

The paper examines in some detail the dependence of the early purchases on the parameters of the problem such as the parameters of the production function and of the stochastic distribution of the rations made available to the enterprise. Certain particular problems are also considered, namely the effect of defective inputs and of output uncertainty. In the most general case there are three sources of uncertainty in the model: uncertainty due to rationing, uncertainty in production and uncertainty as to the proportion of defective inputs. Both manipulable and nonmanipulable rationing mechanisms are investigated.
Output Targets, Input Rationing and Inventories*

by

Stephen M. Goldfeld
Richard E. Quandt

1. Introduction

The enterprise or firm in a socialist planned economy may frequently find itself constrained in two ways: on the one hand it is obligated to produce an output level that is as close as possible to a target level \( Q^* \), and on the other hand it may find that it cannot obtain all the inputs it desires for production. Since inputs are not necessarily rationed at all times, the firm may wish to purchase inputs earlier than would be necessary in the absence of rationing if at the earlier time they are available with no restrictions. Kornai (1979, 1980, 1985) has argued that such anticipatory purchases exacerbate conditions of shortage since even in periods in which the supply of inputs would otherwise be plentiful, firms will make anticipatory purchases and eliminate the potential surplus of the input commodity in question.

It is particularly relevant to examine what happens to this mechanism if the quality of inputs is variable. It is variously noted (Kornai (1980), p.37; (1982), p.29) that chronic shortage leads to a deterioration of output quality, since the shortage causes potential buyers to accept any goods. Substandard input quality may act as if it were another form of rationing, which may then further intensify anticipatory purchases. Hence, the logic of central planning suggests the existence of a self-reinforcing mechanism that exacerbates shortages.

*We are indebted to the National Science Foundation and the National Council for Soviet and East European Research for support.
In Section 2 we outline a model to analyze these questions and in Section 3 we perform some numerical experiments. Section 4 deals with some extensions of the analysis. Section 5 contains brief conclusions.

2. A Theoretical Model

The actions of the manager in our model take place on two discrete dates, namely at time 0 and time 1. Production occurs at time 1 instantaneously with whatever inputs are at hand. The central planning authority gives the manager a target output level $Q^*$ and the enterprise has a strictly concave production function $Q = f(y)$ with a single input $y$; his target level of inputs is $y^*$ given by $Q^* = f(y^*)$. He is penalized for producing an output level less than $Q^*$ and we assume that the penalty is quadratic in the deviation from $Q^*$.

With the exception of one case in Section 4, the manager has no incentive to produce more than $Q^*$ in our set-up which will lead to a one-sided loss function. The manager can order the input $y$ on any of the two dates. There is no limitation on the amount that he can receive on date 0 and the amount of the input received on that date is $y_0$. However, there is a carrying charge or inventory cost of $c_1$ per unit of input obtained at time 0. He may also order an amount $y_1$ on date 1; however, rationing may be in effect on that date and the amount he receives is a random variable.

**The Basic Case.** We assume that the amount of the input that he receives on date 1 is a random variable $x$, with pdf $h(x)$, and is uninfluenced by his desire for a quantity $y_1$. It is convenient to assume that $h(x)$ has support $(0,\infty)$. The total input that will be available is

---

1On managerial incentives see Davis and Charemza (1989).
\[ y^* = y_0 + y_1 \quad \text{if} \quad y_1 < x \]
\[ y_0 + x \quad \text{if} \quad y_1 > x \]

It follows immediately that the expected cost function is

\[
E(C) = c_1y_0 + \int_0^{y^* - y_0} 2c_2[f(x+y_0) - Q^*]^2 h(x)dx \quad (2.1)
\]

where the first term is the inventory cost and the second term is the penalty for missing the target. The objective of the manager is to minimize (2.1) with respect to \( y_0 \).\(^2\) It should be noted that neither the revenues received from the sale of output, nor the cost of the inputs appears in the objective function. We are implicitly assuming that input and output prices are set administratively and that any discrepancy between revenues and costs is eliminated by taxes or subsidies.\(^3\) The first order condition is

\[
\frac{dE(C)}{dy_0} = \int_0^{y^* - y_0} 2c_2[f(x+y_0) - Q^*]f'(x+y_0)h(x)dx - c_2[f(y^*) - Q^*]^2 h(y^* - y_0) + c_1 = 0 \quad (2.2)
\]

where the second term is obviously zero, and the second order condition is

\[
\frac{d^2E(C)}{dy_0^2} = 2c_2 \int_0^{y^* - y_0} \left[ f''(x+y_0) + (f(x+y_0) - Q^*)f''(x+y_0) \right] h(x)dx \quad (2.3)
\]

\(^2\) This is the only variable he needs to choose since \( Q^* \) and hence \( y^* \) are given and he would never order an amount \( y_1 \) greater than \( y^* - y_0 \).

\(^3\) This, in itself, creates a novel incentive structure with which we shall not deal here. See Goldfeld and Quandt (1988).
Proposition 1: The second order condition for a minimum is always satisfied.

Proof: By the definitions of \( y^* \) and \( Q^* \), \( f(x + y_0) \leq Q^* \). The result then follows from the concavity of the production function.

Clearly, optimization of \( E(C) \) is subject to the constraint \( y_0 \neq 0 \) and if the unconstrained solution of (2.2) yields a negative \( y_0 \), then the constrained, economically meaningful solution is \( y_0 = 0 \) (since \( E(C) \) is strictly convex, we need not search for another interior solution).

The Case of Defective Inputs. We now consider a model that is similar to the basic case in all respects except the realistic modification that only a fraction \( \varepsilon \) of inputs ordered is of standard (good) quality, so that a fraction \( 1 - \varepsilon \) is defective and thus unusable.\(^4\) For simplicity, we assume that the same value of \( \varepsilon \) applies to both \( y_0 \) and \( y_1 \); this is tantamount to assuming that at time zero the manager has to decide on \( y_0 \) on the basis of the knowledge he has about \( \varepsilon \) and in the belief that this knowledge is also applicable to \( y_1 \).

At time 0 the manager acquires \( y_0 \) units of the input of which \( \varepsilon y_0 \) units are usable. He will then need \( y^*_\varepsilon - \varepsilon y_0 \) good units at time 1 and hence he has to order \( (y^*_\varepsilon - \varepsilon y_0)/\varepsilon \) units to be assured of an adequate supply of good ones. In fact, he receives an amount which is the \( \min(x, (y^*_\varepsilon - \varepsilon y_0)/\varepsilon) \). It follows that the expected cost function is

\[
E(C_{t\varepsilon}) = c_1\varepsilon y_0 + \int_0^{y^*_\varepsilon / \varepsilon - y_0} c_2 \left[ f(\varepsilon(x + y_0)) - Q^* \right]^2 h(x) dx
\]

\( (2.4) \)

\(^4\)We assume that any defective inputs can be costlessly disposed of so that they do not need to be inventoried.
If $\varepsilon$ is itself a random variable on $(0,1)$ with pdf $g(\varepsilon)$, the expected cost function to be minimized is

$$E(C) = \int_{0}^{1} E(C|\varepsilon) g(\varepsilon) d\varepsilon$$  \hfill (2.5)$$

The mechanism we envisage here is therefore the following. The manager knows that $\varepsilon$ is a random variable with pdf $g(\varepsilon)$, but he also knows that $\varepsilon$ is not resampled every time he orders; rather it is characteristic of the supplier of inputs he uses. Thus, whatever $\varepsilon$ is drawn when he receives $y_0$, it is the same $\varepsilon$ that characterizes the order at time $1$.\(^5\)

A somewhat simpler version of (2.5) is obtained if we assume that $\varepsilon$ can attain only two values: 1.0, in which case all units of inputs are good, and $\varepsilon_0 < 1$, where these two outcomes occur with probabilities $p$ and $1-p$ respectively. The expected cost then is

$$E(C) = pE(C|1) + (1-p) E(C|\varepsilon_0)$$ \hfill (2.6)$$

As before, there is no guarantee that the optimal solution to $y_0$ is positive. Also, the comparative statics results of interest are difficult to derive and we investigate these questions in more detail in Section 3.

**An Example.** For the rest of this section, we consider a simplified example in order to illustrate the solution and to analyze the comparative statics of the model. In this example, we assume that the production function is $f(y) = y$ and that $h(x) = \beta e^{-\beta x}$, the exponential density.\(^6\) At first we assume that there are no defective inputs. We equate the first derivative of $E(C)$ to zero which yields

---

\(^5\)This assumption is relaxed in Section 4.

\(^6\)This form of rationing is chosen for computational convenience and is not intended to be realistic.
\( y^* - y_o + \frac{1}{\beta} \left[ e^{-\beta(y^*-y_o)} - 1 \right] = \frac{c_1}{2c_2} \) \hspace{1cm} (2.7)

The left hand side is zero at \( y_o = y^* \), is arbitrarily large when \( y_o \) is arbitrarily small and has negative slope for all \( y_o < y^* \); the right hand side is a constant. Hence a unique solution exists. Moreover, this solution will be positive if, at \( y_o = 0 \), the left hand side exceeds the right. That will be the case if and only if \( y^* + \frac{(\exp(-\beta y^*) - 1)}{\beta} > \frac{c_1}{2c_2} \).

The interpretation of this condition is straightforward. A positive optimal \( y_o \) is likely to exist if the inventory cost is small relative to the penalty of underproducing. Since this likely to be the case in a centrally planned economy, it is plausible that the expectation of rationing actually induces anticipatory input purchases. The condition is likely to fail when \( \beta \) is very small (it is easy to verify that the limit of the left hand side of the condition as \( \beta \to 0 \) is zero). The reason for this is that as \( \beta \to 0 \) the probability that rationing will occur itself goes to zero and hence positive anticipatory purchases make no sense.

Assume now that the condition for a unique positive solution holds. Then we have the following comparative statics results: \( \frac{\partial y_o}{\partial y^*} = 1 \), \( \frac{\partial y_o}{\partial \beta} > 0 \), \( \frac{\partial y_o}{\partial c_1} < 0 \), \( \frac{\partial y_o}{\partial c_2} > 0 \). To see these results we take the total differential of the first order condition obtaining

\[
\left[ 1 - e^{-\beta(y^*-y_o)} \right] dy^* - \left[ 1 - e^{-\beta(y^*-y_o)} \right] dy_o +
\]

\[
\left[ - \frac{1}{\beta^2} \left( e^{-\beta(y^*-y_o)} - 1 \right) + \frac{1}{\beta} e^{-\beta(y^*-y_o)} (y_o - y^*) \right] d\beta = \frac{1}{2c_2} \cdot dc_1 - \frac{c_1}{2c_2} \cdot dc_2
\]

(2.8)

The results for \( \frac{\partial y_o}{\partial y^*} \), \( \frac{\partial y_o}{\partial c_1} \), \( \frac{\partial y_o}{\partial c_2} \) follow trivially and are obviously
sensible. To get the result for $\partial y_o/\partial \beta$ note that the coefficient of $d\beta$ in (2.8) is positive if $e^{\beta(y^*-y_o)} > 1 + \beta(y^*-y_o)$, which it obviously is from the Taylor series expansion of $e^{\beta(y^*-y_o)}$. A large value of $\beta$ implies that large input allocations are improbable; hence all the results are clearly reasonable.

The same example can be analyzed in the case of defective inputs. In particular, if we restrict attention to the two-point distribution used in (2.6), the relevant first order condition, analogous to (2.7), is given by

$$
p[(y^*-y_o) + \frac{1}{\beta} (e^{-\beta(y^*-y_o)} -1)] + (1-p)[\frac{2}{c_o}(\frac{y^*}{c_o} - y_o + \frac{1}{\beta} (e^{-\beta(\frac{y^*}{c_o} - y_o)} -1))]
$$

$$
= \frac{c_1}{2c_2} [p + (1-p)\epsilon_o] \quad (2.9)
$$

As in the case of (2.7), there are a set of restrictions on the parameters (including $p$ and $\epsilon_o$) that assure that (2.9) yields a unique positive solution for $y_o$. Some further restrictions will assure that the optimal $y_o$ is less than $y^*$. In this range, total differentiation of (2.9) yields the following comparative statics results: $\partial y_o/\partial y^* > 1$, $\partial y_o/\partial \beta > 0$, $\partial y_o/\partial c_1 < 0$, and $\partial y_o/\partial c_2 > 0$. With the exception of the first of these, the results are identical to those in the no-defective case. More general comparative statics results are difficult to obtain analytically, even in this simple example. Consequently when, in the next section, we consider a more realistic example, we shall analyze the behavior of the optimal $y_o$ by numerical methods.

3. The Defective-Input Case Explored Further

In the present section, we shall compute the optimal $y_o$ level for the defective-input model with expected cost function (2.6) under various
assumptions.\textsuperscript{7} The production function is taken to be

\[ Q = y^\alpha \]  

(3.1)

and the distribution of \( x \) is truncated normal, so that

\[ h(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}/\Pr\{x>0\} \]  

(3.2)

The parameters that can be varied are \( c_1, c_2, Q^*, \varepsilon_0, p, \alpha, \mu \) and \( \sigma^2 \). We vary the parameters one at a time, holding all others at their "base case" values which are given in Table 3.1.

\begin{table}[h]
\centering
\begin{tabular}{l}
\hline
\textbf{Table 3.1 Base Case Parameter Values} \\
\hline
\( c_1 = \) & 1.0 \\
\( c_2 = \) & 0.1 \\
\( Q^* = \) & 100.0 \\
\( \varepsilon_0 = \) & 0.7 \\
\( p = \) & 0.9 \\
\( \alpha = \) & 0.8 \\
\( \mu = \) & 200.0 \\
\( \sigma^2 = \) & 50.0 \\
\hline
\end{tabular}
\end{table}

It should be noted that \( \mu \) and \( \sigma^2 \) refer to the parameters of the normal density which gets truncated; the mean and variance of the truncated density are 219.95 and 45.85 respectively. At the base case values the optimal \( y_0 = 55.2 \). The total expected cost is 106.467. Table 3.2 displays the optimal \( y_0 \) levels for sundry variations of the parameters; these are kept at their base values except the parameter named at the top of each column which varies in equal increments as we move down that column from the lowest to the highest value shown in parentheses. For this reason one particular row in each column

\textsuperscript{7}Numerical optimization employed the GRADX algorithm of the GQOPT4 package.
reproduces the base case; this occurs in row six of each column except column 3 where it occurs in the last row.

<table>
<thead>
<tr>
<th>$Q^*$</th>
<th>$\zeta_0$</th>
<th>$p$</th>
<th>$\alpha$</th>
<th>$c_1$</th>
<th>$\mu$</th>
<th>$\sigma^2$</th>
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<td>(75.0-</td>
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<td>(0.45-</td>
<td>(0.75-</td>
<td>(0.75-</td>
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<td>(25.0-</td>
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<td>120.0)</td>
<td>0.90)</td>
<td>0.90)</td>
<td>0.84)</td>
<td>1.20)</td>
<td>240.0)</td>
<td>70.0)</td>
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<td>50.16</td>
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<td>52.69</td>
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<td>58.96</td>
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</tr>
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<td>55.24</td>
<td>70.10</td>
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<td>55.24</td>
<td>55.24</td>
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<td>66.06</td>
<td>44.81</td>
<td>51.59</td>
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<td>130.26</td>
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<td>12.86</td>
<td>41.09</td>
<td>15.25</td>
<td>62.56</td>
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</tbody>
</table>

The comparative statics behavior of $y_0$ is qualitatively as expected. An increase in $Q^*$ increases the anticipatory buying of $y_0$. In fact, for $Q^* = 75, 80$, no $y_0$ is necessary. For higher values of $Q^*$, the quantity of good inputs needed rises absolutely and in percentage terms (for $\alpha = 0.8, y^* = 316.23$ when $Q^* = 100$, and $y^* = 397.17$ when $Q^* = 120$, resulting in a near doubling of inputs bought early as a percentage of the good inputs needed). Both the increase in $\zeta_0$ and in $p$ increase the expected fraction of good inputs and the effect of increasing either results in a decline in the quantity of $y_0$.

The joint effect on $y_0$ of variations in $p$ and $\zeta_0$ can perhaps best be seen by computing for each $p$ and $\zeta_0$ combination the expected fraction $m$ of good inputs ($m = p + (1-p)\zeta_0$) and the standard deviation of the fraction of good inputs $s = (1-\zeta_0)[p(1-p)]^{1/2}$ and computing a response surface by regressing $y_0$ on $m$ and $s$. This yields the regression equation
\[ y_0 = 361.159 - 310.614m - 57.076s \]
\[ (103.364) \quad (-92.450) \quad (-14.080) \]

where \( t \) values are in parenthesis and \( R^2 = 0.9987 \). Both an increase in the mean and the standard deviation significantly reduce \( y_0 \).

The effect of increasing \( a \) is to diminish \( y_0 \). Clearly, the larger is \( a \), the smaller is the \( y^* \) needed to produce \( Q^* \) and hence the smaller will \( y_0 \) be. As \( a \) increases, not only does \( y_0 \) decrease, but it also decreases as a fraction of \( y^* \) (from 0.27 = 128.64/464.16 to 0.053 = 12.86/240.41). An increase in \( c_1 \) relative to \( c_2 \) makes carrying inventories more expensive and reduces \( y_0 \). The (arc) elasticity of \( y_0 \) with respect to \( c_1 \) is increasing as \( y_0 \) falls, namely from 0.89 to 1.96. Finally, as \( \mu \) increases or as \( \sigma^2 \) declines, \( y_0 \) falls. This is to be expected since both changes have the effect of reducing the probability that the manager will be rationed at all. Here again we can employ the response surface methodology. We regress \( y_0 \) on the true mean and variance of the truncated normal which are implied by \( \mu \) and \( \sigma^2 \). Denoting these by \( \mu_t \) and \( \sigma^2_t \) respectively, we have

\[ y_0 = 241.020 - 0.995\mu_t + 0.7290\sigma^2_t \]
\[ (153.191) \quad (-140.618) \quad (46.466) \]

with an \( R^2 = 0.9991 \), which clearly tells the same story.

While these examples are only illustrative, it is noteworthy that the effects are quite pronounced and that very substantial anticipatory buying occurs either if the probability of rationing is high or if the expected number of defective inputs is high. The qualitative results are not contingent on the assumption of truncated normality because similar results are obtained if the Weibull distribution is employed.
4. Extensions

We consider three modifications of the models discussed earlier.

**Production Uncertainty.** It is plausible to argue that the outcome of the production process is not known with certainty by the manager (Goldfeld and Quandt (1988)). The production function is then written as

\[ Q = f(y)e^u \]  

(4.1)

where \( u \) is a random variable, and in the first instance it is reasonable to assume that \( u \sim N(0, \omega^2) \). Denoting the normal density by \( n(0, \omega^2) \), (2.4) is then replaced by

\[
E(C) \mid \epsilon, u = c_1 \epsilon y_0 + \int_{0}^{\infty} c_2 \left[ f(\epsilon(x+y_0))e^{u-Q}\right]^2 h(x)n(0, \omega^2)dx 
\]

(4.2)

and \( E(C) \mid \epsilon \) becomes

\[
E(C) \mid \epsilon = c_1 \epsilon y_0 + \int_{0}^{\infty} \int_{0}^{\infty} c_2 \left[ f(\epsilon(x+y_0))e^{u-Q}\right]^2 h(x)n(0, \omega^2)dx du
\]

\[
= c_1 \epsilon y_0 + \int_{0}^{\infty} c_2 \left[ f(\epsilon(x+y_0))e^{Q^2} + Q^2 - 2f(\epsilon(x+y_0))Q e^{2\omega^2/2}\right]h(x)dx
\]

(4.3)

In order to choose plausible values for \( \omega^2 \) we assume that \( f(\epsilon)e^u \) differs from \( f(\epsilon) \) 95 percent of the time by no more than 5 (or 10) percent. This leads to \( \omega^2 \) values of 0.00065 and 0.0026 respectively. We repeat only the experiment in which \( Q^2 \) varies, with the remaining parameters set to base case values. The values of \( y_0 \) are in Table 4.1.
Two interesting features emerge. The first one is that the sensitivity of $y_0$ to modest production uncertainty is slight in absolute terms; thus, for example a production uncertainty that changes output by no more than 10 percent 95 percent of the time alters the base case $y_0$ from 55.24 to 54.96, i.e., less than 1/2 percent. The second one is that the direction of the effect is not uniform: at relatively low levels of $Q^*$, production uncertainty reduces the optimal $y_0$ while at higher levels of $Q^*$ it increases $y_0$. We suggest the following intuition for this phenomenon. At low $Q^*$ levels the probability that the enterprise will be rationed is small and over or underproduction is not very costly (since stochastic deviation from $f(\epsilon(x+y_0))$ is multiplicative). Hence, it makes sense to save on inventory costs by reducing $y_0$, particularly because stochastic overproduction can (partially) compensate for input rationing. However, at high $Q^*$ values rationing is likelier and the penalty for underproduction increases, which suggests that $y_0$ ought to increase.

**Independent Distributions of Defective Inputs.** In the previous sections it was assumed that once $\epsilon$ is drawn, it applies equally to time 0 and time 1. While this is not necessarily a bad assumption (the enterprise may deal repeatedly with the same supplier of inputs who may not effect significant

Table 4.1 Effect on $y_0$ of $Q^*$ Variations

<table>
<thead>
<tr>
<th>$Q^*$</th>
<th>$\omega^2=0.00065$</th>
<th>$\omega^2=0.0026$</th>
</tr>
</thead>
<tbody>
<tr>
<td>75.0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>80.0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>85.0</td>
<td>2.51</td>
<td>2.34</td>
</tr>
<tr>
<td>90.0</td>
<td>19.96</td>
<td>19.80</td>
</tr>
<tr>
<td>95.0</td>
<td>37.58</td>
<td>37.45</td>
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<td>105.0</td>
<td>73.28</td>
<td>73.30</td>
</tr>
<tr>
<td>110.0</td>
<td>92.25</td>
<td>92.35</td>
</tr>
<tr>
<td>115.0</td>
<td>111.82</td>
<td>111.98</td>
</tr>
<tr>
<td>120.0</td>
<td>130.26</td>
<td>130.41</td>
</tr>
</tbody>
</table>
changes over the short run in his own production processes), it is clearly only one of several possible scenarios. A polar opposite to this assumption of perfect correlation between the defective rates on the two dates is complete independence.  

To implement this assumption, we assume specifically that at time 0 the fraction of "good" inputs is 1.0 or ε₀ with probability p₀ and 1-p₀ respectively and that at time 1 the corresponding magnitudes are ε₁ and p₁. A slight generalization of (2.6) yields for expected cost

$$E(C) = \int_0^{y^*-\epsilon_0 y_0} c_2 \left[ f(y_0 + x) - Q^* \right]^2 h(x) dx + p_0 (1-p_1) \int_0^{y^*-\epsilon_1 x} c_2 \left[ f(y_0 + \epsilon_1 x) - Q^* \right]^2 h(x) dx$$

$$+ (1-p_0)p_1 \int_0^{y^*-\epsilon_0 y_0} c_2 \left[ f(\epsilon_0 y_0 + x) - Q^* \right]^2 h(x) dx + c_1 y_0 (p_0 + (1-p_0) \epsilon_0) $$

We report only two experiments in which we minimize (4.4) for various Q* levels (see Table 4.2). In the first, ε₁=0.7, p₁=0.9; the same values as we use for ε₀, p₀ respectively. This is the case in which uncertainty concerning defective inputs is heightened and, as one might expect, y₀ increases throughout as compared with Table 3.2. In the other case, p₁=1.0 and this case corresponds to no uncertainty about defective inputs at time 1.

---

8An interesting possibility among "intermediate" scenarios is that the manager employs some prior density for ε at time 0 and employs for time 1 the posterior density based on the realization at time 0.
The obvious consequence, which can be seen from Tables 3.2 and 4.2, is that $y_0$ diminishes.

Table 4.2  Effect on $y_0$ of $\varepsilon_1$, $p_1$

<table>
<thead>
<tr>
<th>$Q^*$</th>
<th>$\varepsilon_0=0.7$</th>
<th>$\varepsilon_1=0.7$</th>
<th>$\varepsilon_0=0.7$</th>
<th>$\varepsilon_1=--$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$p_0=0.9$</td>
<td>$p_1=0.9$</td>
<td>$p_0=0.9$</td>
<td>$p_1=1.0$</td>
</tr>
<tr>
<td>75.0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>80.0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>85.0</td>
<td>4.35</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>90.0</td>
<td>21.69</td>
<td>14.93</td>
<td>32.45</td>
<td>50.05</td>
</tr>
<tr>
<td>95.0</td>
<td>39.20</td>
<td>32.45</td>
<td>50.05</td>
<td>50.05</td>
</tr>
<tr>
<td>100.0</td>
<td>56.76</td>
<td>50.05</td>
<td>68.07</td>
<td>68.07</td>
</tr>
<tr>
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<td>74.73</td>
<td>68.07</td>
<td>87.10</td>
<td>87.10</td>
</tr>
<tr>
<td>110.0</td>
<td>93.63</td>
<td>87.10</td>
<td>107.09</td>
<td>107.09</td>
</tr>
<tr>
<td>115.0</td>
<td>113.29</td>
<td>107.09</td>
<td>126.27</td>
<td>126.27</td>
</tr>
<tr>
<td>120.0</td>
<td>132.35</td>
<td>126.27</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Manipulable Rationing.** An alternative form of rationing takes place when the amount of the input received is influenced by the amount ordered by the firm. The simplest and most plausible way this can occur is if, when $y_1$ is ordered at time 1, the manager receives $x y_1$ where $x$ is a random variable distributed over the $(0,1)$ interval. In this case the penalty function must be symmetric, i.e., overproduction must also be penalized; otherwise the manager would have no incentive to order a finite amount.

The expected cost function is

$$E(C) = \int_0^1 c_2 \left[f(y_0+xy_1) - Q^*\right]^2 h(x)dx + c_1 y_0$$

(4.5)

The first order conditions for a minimum are
\[
\frac{\partial E(c)}{\partial y_0} = 2c_2 \left[ \int_0^1 [f - Q^*] f' h(x) \, dx + c_1 \right] = 0 \tag{4.6}
\]

\[
\frac{\partial E(c)}{\partial y_1} = 2c_2 \left[ \int_0^1 [f - Q^*] f' x h(x) \, dx \right] = 0 \tag{4.7}
\]

where \( f \) and \( f' \) have \( y_0 + xy_1 \) as their argument. Taking the total differential of (4.6) and (4.7) and solving for \( \frac{\partial y_0}{\partial c_1} \) and \( \frac{\partial y_0}{\partial c_2} \), we obtain

\[
\frac{\partial y_0}{\partial c_1} = \frac{-\frac{2}{\partial E(c)/\partial y_1^2}}{D} \tag{4.8}
\]

\[
\frac{\partial y_0}{\partial c_2} = \frac{-2 \left[ \int_0^1 [f - Q^*] f' h(x) \, dx (\frac{2}{\partial E(c)/\partial y_1}) + 2 \left[ \int_0^1 [f - Q^*] f' x h(x) \, dx (\frac{2}{\partial E(c)/\partial y_0 y_1}) \right] \right]}{D} \tag{4.9}
\]

where \( D \) is the determinant of the Hessian matrix. If the second order conditions hold, \( \frac{\partial y_0}{\partial c_1} < 0 \). Further, the second integral in the numerator of (4.9) is zero by (4.7) and the first integral is \( -c_1/2c_2 \) by (4.6); hence \( \frac{\partial y_0}{\partial c_2} > 0 \). None of the other comparative statics derivatives is given unambiguously and we again resort to a simplified example.

It is easy to show that on the same simplifying assumption as before, that the production function is linear, and on the further assumption that \( h(x) \) is uniform, \( E(C) \) is given by

\[
E(C) = c_2(y_0 - Q^*)^2 + c_2y_1(y_0 - Q^*) + c_2y_1^2/3 + c_1y_0 \tag{4.10}
\]

The manager now has to decide on both \( y_0 \) and \( y_1 \). Setting the first partial derivates of \( E(c) \) equal to zero yields
\[ y_0 = Q^* - 2c_1/c_2 \]
\[ y_1 = 3c_1/c_2 \]

This provides the solution if and only if \( Q^* \leq 2c_1/c_2 \); otherwise the optimal solution is obtained by setting \( y_0 = 0 \) and minimizing (4.10) with respect to \( y_1 \) alone, which yields \( y_1 = 3Q^*/2 \).\(^9\) The qualitative result from (4.11) is that the optimal \( y_0 \) declines as \( c_1 \), the inventory cost, increases relative to \( c_2 \). The total expected input use, \( y_0 + E(x)\ y_1 \), equals \( Q^* - c_1/2c_2 \), while the total amount ordered, \( y_0 + y_1 \), equals \( Q^* + c_1/c_2 \) which exceeds \( y^* \), the amount required to produce the target output level.

In the case of defective inputs we replace \( y_0 \) and \( y_1 \) in (4.10) by \( \varepsilon y_0 \) and \( \varepsilon y_1 \) respectively, and assume that \( \varepsilon \) is uniformly distributed on \((0,1)\). Equation (4.10) can then be shown to be

\[ E(C) = c_2\left[ \frac{y_0^2}{3} + Q^*2 - y_0Q^* + \frac{y_0y_1}{3} - \frac{y_1Q^*}{2} + \frac{y_1^2}{9} \right] + \frac{c_1y_0}{2}. \]

Setting the partial derivatives equal to zero yields \( y_0 = 3Q^*/2 - 3c_1/c_2 \), \( y_1 = 9c_1/2c_2 \). This is the solution if \( Q^* \leq 2c_1/c_2 \). The expected input use is \( y_0 + E(x)y_1 = (3/2)(Q^* - c_1/2c_2) \), which is 50 percent greater than in the case with no defective inputs. This increase is less than the doubling that might be suggested by the mean defective rate of .5. This attenuation obviously stems from the symmetric quadratic penalties embodied in the objective function, (4.5).

\(^9\)It is obvious that the second order conditions hold.
5. Conclusions

We formulated several related models in which an input is subject to rationing and may be defective. We have considered both manipulable and nonmanipulable rationing and treated in some detail variations in three sources of uncertainty: the uncertainty of the rationing process, the uncertainty of the production process itself, and that pertaining to the fraction of inputs that turn out to be defective. Since the enterprise is able to purchase the input at an earlier time, we concentrated on the quantity $y_0$ of early purchases which, however, incur a carrying cost. While some analytic results are available, much of the comparative statics analysis was based on numerical computations with illustrative functions. In all cases, the behavior of $y_0$ appeared sensible.

Some interesting questions remain for future research and we mention only four areas. (1) To the extent that both $Q^*$ and the rationing of the input are set by the central planning authority, it may be plausible to argue that the density of the input ration $x$ should be explicitly conditional on $Q^*$; thus $h(x)$ would have to be replaced everywhere by $h(x|Q^*)$. (2) The analysis may be extended to two or more inputs, which would allow an examination of the dependence of $y_0$ on the elasticity of substitution between inputs. (3) As mentioned earlier, the manager may perform a Bayesian update and employ for the density of the fraction of defectives at time 1 the appropriate posterior density. Finally, (4) the model is limited due to its two-period nature. It would be desirable to provide a full dynamic formulation in which inputs can be stored for more distant periods in the future. We hope to shed light on these questions in future work.
References


——— (1980), The Economics of Shortage, Amsterdam: North-Holland.